(max, min)-convolution and mathematical morphology

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Abstract. A formal definition of morphological operators in $(\max, \min)$-algebra is introduced and their relevant properties from an algebraic viewpoint are stated. Some previous works in mathematical morphology have already encountered this type of operators but a systematic study of them has not yet been undertaken in the morphological literature. It is shown in particular that one of their fundamental property is the equivalence with level set processing using Minkowski addition and subtraction. Theory of viscosity solutions of the Hamilton-Jacobi equation with Hamiltonians containing u and Du is summarized, in particular, the corresponding Hopf-Lax-Oleinik formulas as $(\max, \min)$-operators. Links between $(\max, \min)$-convolutions and some previous approaches of unconventional morphology, in particular fuzzy morphology and viscous morphology, are reviewed.

Keywords: Minkowski addition; adjunction; Hamilton–Jacobi PDE; fuzzy morphology; viscous morphology

1 Introduction

Let E be the Euclidean $\mathbb{R}^n$ or discrete space $\mathbb{Z}^n$ (support space) and let $\mathcal{T}$ be a set of grey-levels (space of values). For theoretical reasons it is typically assumed that $\mathcal{T} = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, but one often has $\mathcal{T} = [0, M]$. A grey-level image is represented by a function $f \colon E \to \mathcal{T}$, also noted as $f \in \mathcal{F}(E, \overline{\mathbb{R}})$, such that f maps each pixel $x \in E$ into a grey-level value in $\mathcal{T}$. Given a grey-level image, the two basic morphological mappings $\mathcal{F}(E, \mathcal{T}) \to \mathcal{F}(E, \mathcal{T})$ are the dilation and the erosion given respectively by

$$\begin{cases} (f \oplus b)(x) = \sup_{y \in E} \left\{ f(y) + b(x-y) \right\}, \\ (f \ominus b)(x) = \inf_{y \in E} \left\{ f(y) - b(y-x) \right\}, \end{cases} \tag{1}$$

where $b \in \mathcal{F}(E, \mathcal{T})$ is the structuring function which determines the effect of the operator. The other morphological operators, such as the opening and the closing, are obtained by composition of dilation/erosion [20,11]. The Euclidean framework has been recently generalized to images supported on Riemannian manifolds [2]. Operators (1) can be interpreted in nonlinear mathematics as the convolution in $(\max, +)$-algebra (and in its dual algebra) [10]. This inherent connection of functional operators (1) with the supremal and infimal convolution of nonlinear mathematics and convex analysis has been extremely fruitful to the state-of-the-art on mathematical morphology (morphological PDE, slope transform, etc.). Nevertheless, the functional operators (1) do not extend all the fundamental properties of the dilation and erosion for sets, as formulated in Matheron's theory. Perhaps the most disturbing for us are, on the one hand, the lack of commutation with level set processing for nonflat structuring functions; on the other hand, the limitation of Matheron's axiomatic of granulometry to constant (i.e., flat) functions on a convex domain [12]. In addition, there are some unconventional morphological frameworks, such as the fuzzy morphology [8,14,7] or the viscous morphology [21,22,15] which do not fit in the classical $(\max, +)$-algebra. Actually, the $(\max, +)$ is not the unique possible alternative to see morphological operators as convolutions. The idea in this paper is to consider the operation of convolution of two functions in the $(\max, \min)$-algebra. This is in fact our main motivation: to formally introduce the notion of $(\max, \min)$-mathematical morphology. As we show in the paper, this framework is not totally new in morphology since some fuzzy morphological operators are exactly the same convolutions that we introduce. But some of the key properties are ignored by in the fuzzy context, and the most important, they are not limited to fuzzy sets. By the way, even if much less considered than the supremal and infimal convolutions, convolutions in $(\max, \min)$-algebra have been the object of various studies in different branches of nonlinear applied mathematics, from quasi-convex analysis [24,19,25,9,13,17] to viscosity solutions of Hamilton-Jacobi equations [5,6,1,23]. Interested reader is also referred to the book [10] for a systematic comparative study of matrix algebra and calculus in the three algebras $(+, \times)$, $(\max, +)$ and $(\max, \min)$.

The present work is exclusively a theoretical study and thus the practical interest of the operators is not illustrated here. Complete proofs and additional results can be found in [3].

We use the following representation of semicontinuous functions. Given an upper semicontinuous (USC) function $f \in \mathcal{F}(E, \overline{\mathbb{R}})$, it can be defined by means of its upper level sets $X_h^+(f)$ as follows $f(x) = \sup\left\{h \in \overline{\mathbb{R}} \,:\, x \in X_h^+(f)\right\}$, or by its strict lower level sets $Y_h^-(f)$: $f(x) = \inf\left\{h \in \overline{\mathbb{R}} \,:\, x \in Y_h^-(f)\right\}$, where

$$X_h^+(f) = \{x \in E \,:\, f(x) \geq h\}\,, \text{ and } Y_h^+(f) = \{x \in E \,:\, f(x) > h\}\,;$$
$$X_h^-(f) = \{x \in E \,:\, f(x) \leq h\}\,, \text{ and } Y_h^-(f) = \{x \in E \,:\, f(x) < h\}\,.$$

A continuous function f can be decomposed/reconstructed using either its (strict) upper level sets or its (strict) lower level sets. One has $\left(X_h^+(f)\right)^c = Y_h^-(f)$.

2 $(\max, \min)$-convolutions: definition and properties

In this Section we define the alternative convolutions associated to a pair *(function f, structuring function b)* in the $(\max, \min)$ mathematical framework. We also study their properties.

Definition 1. *Given a structuring function $b \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$, for any function $f \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$ we define the supmin convolution $f \bigtriangledown b$ and the infmax convolution $f \bigtriangleup b$ of f by b as*

$$(f \bigtriangledown b)(x) = \sup_{y \in \mathbb{R}^n} \left\{ f(y) \wedge b(x-y) \right\}, \tag{2}$$

$$(f \bigtriangleup b)(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) \vee b^c(y-x) \right\}. \tag{3}$$

We also define the adjoint infmax $f \bigtriangleup^* b$ and the adjoint supmin $f \bigtriangledown^* b$ convolutions as

$$(f \bigtriangleup^* b)(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) \wedge^* b(y-x) \right\}, \tag{4}$$

$$(f \bigtriangledown^* b)(x) = \sup_{y \in \mathbb{R}^n} \left\{ f(y) \vee^* b^c(x-y) \right\}, \tag{5}$$

where $\wedge^*$ is the adjoint operator to the minimum $\wedge$ and is given by

$$f(y) \wedge^* b(y-x) = \begin{cases} f(y) & \text{if } \; b(y-x) > f(y) \\ \top & \text{if } \; b(y-x) \leq f(y) \end{cases} \tag{6}$$

and $\vee^*$ the adjoint to $\vee$:

$$f(y) \vee^* b^c(x-y) = \begin{cases} f(y) & \text{if } \; b^c(x-y) < f(y) \\ \bot & \text{if } \; b^c(x-y) \geq f(y) \end{cases} \tag{7}$$

and where, if we define $\max g = \sup_{x \in \mathbb{R}^n} g(x)$ and $\min g = \inf_{x \in \mathbb{R}^n} g(x)$, the top and bottom elements for pair of functions f and b correspond to

$$\top = (\max f) \vee (\max b) \quad \text{and} \quad \bot = (\min f) \wedge (\min b^c).$$

Definitions remain valid if we replace $\mathbb{R}^n$ by a subset E or any subset of discrete space $\mathbb{Z}^n$. Similarly, the extended real line $\overline{\mathbb{R}}$ can be replaced by a bounded, eventually discrete, set of intensities $[0, M]$. Figure 1 illustrates the four $(\max, \min)$-convolutions for a given example of one dimensional functions defined in a bounded interval, i.e., $f, b \in \mathcal{F}(\mathbb{R}, [0, M])$.

Duality by complement vs. duality by adjunction. From a morphological viewpoint, their most salient properties are summarized in this proposition (proof in [3]).

Proposition 1. *The supmin convolution $\bigtriangledown$ and infmax convolution $\bigtriangleup$ are dual with respect to the complement. Similarly, the adjoint infmax convolution $\bigtriangleup^*$ and the adjoint supmin $\bigtriangledown^*$ convolution are dual with respect to the complement, i.e., for $f, b \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$ one has*

$$f \bigtriangleup b = \left(f^c \bigtriangledown \check{b} \right)^c \quad \text{and} \quad f \bigtriangledown b = \left(f^c \bigtriangleup \check{b} \right)^c \tag{8}$$

$$f \bigtriangleup^* b = \left(f^c \bigtriangledown^* \check{b} \right)^c \quad \text{and} \quad f \bigtriangledown^* b = \left(f^c \bigtriangleup^* \check{b} \right)^c \tag{9}$$

The pair $(\bigtriangleup^*, \bigtriangledown)$ forms an adjunction. Similarly, the pair $(\bigtriangleup, \bigtriangledown^*)$ is also an adjunction, i.e., for $f, g, b \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$ one has

$$f \bigtriangledown b \leq g \Longleftrightarrow f \leq g \bigtriangleup^* b \tag{10}$$

$$f \bigtriangledown^* b \leq g \Longleftrightarrow f \leq g \bigtriangleup b \tag{11}$$

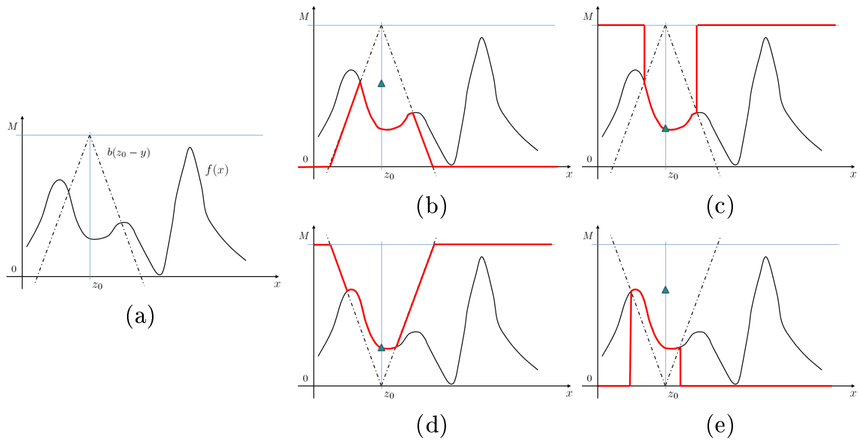

Fig. 1. Illustration of four $(\max, \min)$-convolutions for a given example of one dimensional functions defined in a bounded interval, i.e., $f, b \in \mathcal{F}(\mathbb{R}, [0, M])$: (a) original function $f(x)$ and translated structuring function b at point z_0; (b) in red, $f(y) \wedge b(z_0 - y)$ for all $y \in \mathbb{R}$, green triangle represents $(f \bigtriangledown b)(x)$ the value of the supmin convolution at z_0; (c) in red, $f(y) \wedge^* b(y - z_0)$, green triangle, adjoint infmax at z_0: $(f \bigtriangleup^* b)(z_0)$; (d) in red, $f(y) \vee b^c(y - z_0)$, green triangle, infmax at z_0: $(f \bigtriangleup b)(z_0)$; (e) in red, $f(y) \vee^* b^c(z_0 - y)$, green triangle, adjoint supmin at z_0: $(f \bigtriangledown^* b)(z_0)$.

Commutation with level set processing. We can introduce now the fundamental property of $(\max, \min)$-convolutions (proof in [3]).

Proposition 2. *Let f and b in $\mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$. Then the four $(\max, \min)$-convolutions of f by b obey the following commutation rules of level sets with respect to Minkowski sum and substraction: for all $h \in \overline{\mathbb{R}}$*

$$X_h^+(f \bigtriangledown b) = X_h^+(f) \oplus X_h^+(b) \tag{12}$$
$$Y_h^-(f \bigtriangleup b) = Y_h^-(f) \oplus Y_h^-(\check{b}^c) \tag{13}$$
$$X_h^+(f \bigtriangleup^* b) = X_h^+(f) \ominus X_h^+(b) \tag{14}$$
$$Y_h^-(f \bigtriangledown^* b) = Y_h^-(f) \ominus Y_h^-(\check{b}^c) \tag{15}$$

This expression on strict lower level sets Y_h^- for $(f \bigtriangleup b)$ is valid for lower level sets X_h^- if $(f \bigtriangleup b)$ is *exact*, in the sense that, for each $x \in \mathrm{dom}^-(f \bigtriangleup b)$, there exists $y \in \mathbb{R}^n$ such that $(f \bigtriangleup b)(x) = f(y) \vee b^c(y - x)$ (i.e., the minimum is attained for any x in the domain) [19,13]. In particular, if f and b^c are both LSC quasiconvex functions, $(f \bigtriangleup b)$ and $(f \bigtriangledown^* b)$ are exact, which involves $X_h^-(f \bigtriangleup b) = X_h^-(f) \oplus X_h^-(\check{b}^c)$ and $X_h^-(f \bigtriangledown^* b) = X_h^-(f) \ominus X_h^-(\check{b}^c)$.

We need for the sequel an alternative formulation of the infmax and adjoint supmin convolution in terms respectively of Minkowski subtraction $\ominus$ and addition $\oplus$ of level sets. It is simply based on rewriting the infmax convolution using upper level sets:

$$(f \bigtriangleup b)(x) = \inf\left\{h \in \mathbb{R} \,:\, x \in Y_h^-(f \bigtriangleup b)\right\}$$
$$= \sup\left\{h \in \mathbb{R} \,:\, x \in \left(X_h^+(f) \ominus Y_h^-(b^c)\right)\right\}. \quad (16)$$

Analogously, one obtains the following equivalence for the adjoint supmin convolution:

$$(f \bigtriangledown^* b)(x) = \inf\left\{h \in \mathbb{R} \,:\, x \in \left(Y_h^-(f) \ominus Y_h^-(\check{b}^c)\right)\right\}$$
$$= \sup\left\{h \in \mathbb{R} \,:\, x \in \left(X_h^+(f) \oplus Y_h^-(b^c)\right)\right\}. \quad (17)$$

Therefore, we can write

$$X_h^+(f \bigtriangleup b) = X_h^+(f) \ominus Y_h^-(b^c), \quad (18)$$
$$X_h^+(f \bigtriangledown^* b) = X_h^+(f) \oplus Y_h^-(b^c). \quad (19)$$

Further properties. Other useful properties of $(\max, \min)$-convolutions are proven in [3].

Canonic structuring function. The conic structuring function plays a role similar to the multiscale quadratic structuring function in $(\max, +)$-algebra.

Definition 2. *The multiscale conic structuring function is defined as the canonic structuring function in* $(\max, \min)$*-convolution:*

$$c_\lambda(x) = -\frac{\|x\|}{\lambda}. \quad (20)$$

In order to justify this canonicity, let us consider the upper level sets of $c_\lambda(x)$. First, we remind that a ball of radius centered at point x is given by the set $B_r(x) = \{y \in \mathbb{R}^n \,:\, \|x - y\| \leq r\}$.

Proposition 3. *The canonic structuring function in* $(\max, \min)$*-convolution satisfies the semi-group*

$$(c_\lambda \bigtriangledown c_\mu)(x) = c_{\lambda+\mu}(x). \quad (21)$$

In the case of the L^∞ *metric, a dimension separability is obtained for* $c_\lambda^\infty(x) = -\|x\|_\infty/\lambda$*; i.e., let us denote the coordinates of point as* $x = (x_1, x_2, \cdots, x_n)$ *and by* $c_{\lambda;\,i}(x) = -|x_i|/\lambda$ *the one dimensional conic structuring function, we have*

$$c_\lambda^\infty(x) = (c_{\lambda;\,1} \bigtriangledown c_{\lambda;\,2} \cdots \bigtriangledown c_{\lambda;\,n}). \quad (22)$$

It is easy to see this property. We first note that $X_{-h}^+(c_\lambda) = B_{\lambda h}$. Second, we remind the Minkowski addition of balls: $B_{r_1} \oplus B_{r_2} = B_{r_1+r_2}$. Therefore, one has

$$X_{-h}^+(c_\lambda \bigtriangledown c_\mu) = X_{-h}^+(c_\lambda) \oplus X_{-h}^+(c_\mu) = B_{\lambda h} \oplus B_{\mu h} = B_{(\lambda+\mu)h}.$$

Dimension separability in L^∞ metric is also a consequence of the Minkowski addition of segments. As a consequence of the L^∞ dimension separability, the classical theory of Minkowski decomposition of structuring elements [20].

3 Openings, closings using $(\max, \min)$-convolutions and granulometries

The adjointness of the pairs $(\triangle^*, \triangledown)$ and $(\triangle, \triangledown^*)$ involves that from an algebraic viewpoint both the supmin convolution $\triangledown$ and the adjoint supmin convolution $\triangledown^*$ are a dilation; both the infmax convolution $\triangle$ and the adjoint infmax convolution $\triangle^*$ are an erosion. Therefore, their composition naturally yields openings and closings. Let us be more precise.

Definition 3. *Given any USC function $f \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$, the* $(\max, \min)$*-opening and* $(\max, \min)$*-closing of f by the continuous structuring function $b \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$ are respectively given by*

$$(f \lozenge\, b) = ((f \triangle^* b) \triangledown b)\,, \tag{23}$$

and

$$(f \blacklozenge\, b) = ((f \triangledown^* b) \triangle b)\,, \tag{24}$$

such that their corresponding level sets representations, based on expressions (12), (14), *and* (13), (15), *are given by*

$$\begin{aligned} X_h^+(f \lozenge\, b) &= X_h^+(f \triangle^* b) \oplus X_h^+(b) = \left[X_h^+(f) \ominus X_h^+(b)\right] \oplus X_h^+(b) \\ &= X_h^+(f) \circ X_h^+(b)\,, \end{aligned} \tag{25}$$

$$\begin{aligned} Y_h^-(f \blacklozenge\, b) &= Y_h^-(f \triangledown^* b) \oplus Y_h^-\left(\check{b}^c\right) = \left[Y_h^-(f) \ominus Y_h^-\left(\check{b}^c\right)\right] \oplus Y_h^-\left(\check{b}^c\right) \\ &= Y_h^-(f) \circ Y_h^-\left(\check{b}^c\right). \end{aligned} \tag{26}$$

We note that $(\max, \min)$-opening is defined from adjunction $(\triangle^*, \triangledown)$ whereas $(\max, \min)$-closing from $(\triangle, \triangledown^*)$. We can also switch roles and to formulate the so-called second family of dual $(\max, \min)$-opening and closing as

$$(f \lozenge^* b) = ((f \triangle b) \triangledown^* b)\,, \tag{27}$$

$$(f \blacklozenge^* b) = ((f \triangledown b) \triangle^* b)\,, \tag{28}$$

which has the following equivalent interpretation in terms of level sets:

$$Y_h^-(f \lozenge^* b) = Y_h^-(f) \bullet Y_h^-\left(\check{b}^c\right), \tag{29}$$

$$X_h^+(f \blacklozenge^* b) = X_h^+(f) \bullet X_h^+(b)\,. \tag{30}$$

Besides the duality by complement, classical properties of opening and closing hold in the $(\max, \min)$ framework as a consequence of the adjunction [11]. See details in [3].

The extension of the granulometric theory [16] to the framework of $(\max, +)$-based morphology was deeply studied in [12]. In particular, it was proven that one can build grey-level Euclidean granulometries with a multiscale structuring function if and only if structuring function has a convex compact domain and is constant there (i.e., flat function).

In the case of $(\max, \min)$-openings, we can naturally extend Matheron axiomatic of Euclidean granulometries without the flatness limitation (proof in in [3]).

Proposition 4. *Given a structuring function* $b_1 \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$ *such that all its upper level sets* $X_h^+(b_1)$ *are convex sets, the family of multi-scale* $(\max, \min)$*-openings* $\{f \lozenge b_\lambda\}_{\lambda \geq 1}$*, where the structuring function at scale* λ *is given by*

$$b_\lambda(x) = b_1\left(\lambda^{-1}x\right),$$

forms an Euclidean granulometry on any image $f \in \mathcal{F}(\mathbb{R}^n, \overline{\mathbb{R}})$*, i.e.,*

$$(f \lozenge b_\lambda) = \lambda \star \left(\left(\lambda^{-1} \star f\right) \lozenge b_1\right), \tag{31}$$

which involves compatibility with scaling in the spatial domain, in the sense of Matheron's axiomatic defined as follows

$$(\lambda \star f)(x) = f\left(\lambda^{-1}x\right), \ \forall \lambda \geq 1.$$

In addition, we have the following semi-group properties, $\forall \lambda_1, \lambda_2 \geq 1$

$$b_{\lambda_1+\lambda_2}(x) = (b_{\lambda_1} \bigtriangledown b_{\lambda_2})(x), \tag{32}$$

$$((f \lozenge b_{\lambda_1}) \lozenge b_{\lambda_2})(x) = ((f \lozenge b_{\lambda_2}) \lozenge b_{\lambda_1})(x) = \left(f \lozenge b_{\sup(\lambda_1,\lambda_2)}\right)(x) \tag{33}$$

A good candidate of multi-scale isotropic structuring function leading to $(\max, \min)$ granulometries is based on the canonic structuring function (21), as $b_\lambda(x) = c_\lambda(x) + \alpha$, which is equivalent to $b_\lambda(x) = \lambda^{-1}c_1(x) + \alpha$, $\lambda \geq 1$, $\alpha > 0$.

4 Hopf-Lax-Oleinik formulas for Hamilton-Jacobi equation $u_t \pm H(u, Du) = 0$

We study now the Hopf-Lax-Oleinik type formulas for Hamilton-Jacobi PDE of form $u_t \pm H(u, Du) = 0$ and its links to convolutions in $(\max, \min)$-algebra. The theory of this equation was developed by Barron, Jensen and Liu [5,6]. Other interesting results can be found in paper by Alvarez, Barron and Ishii [1] and the excellent survey paper by Van and Son [23]. The most relevant elements for us can be summarized in the following result.

Proposition 5. *Let us consider the two following Cauchy problems (first-order Hamilton-Jacobi PDEs):*

$$\begin{cases} u_t + H_1(u, Du) = 0, \ in \ (x,t) \in \mathbb{R}^n \times (0, \infty), \\ u(x,0) = f(x), \qquad \forall x \in \mathbb{R}^n, \end{cases} \tag{34}$$

and

$$\begin{cases} u_t + H_2(u, Du) = 0, \ in \ (x,t) \in \mathbb{R}^n \times (0, \infty), \\ u(x,0) = g(x), \qquad \forall x \in \mathbb{R}^n, \end{cases} \tag{35}$$

where the initial conditions are functions $f, g : \mathbb{R}^n \times \mathbb{R}$*, such that* f *is a LSC proper function, bounded from below; and* g *an USC proper function, bounded from above. The Hamiltonians* $H_1, H_2 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *are assumed to satisfy the following conditions:*

(A1) *$H_1(\gamma,p)$ and $H_2(\gamma,p)$ are continuous;*
(A2) *$H_1(\gamma,p)$ and $H_2(\gamma,p)$ are nondecreasing in $\gamma \in \mathbb{R}$, $\forall p \in \mathbb{R}^n$;*
(A3) *$H_1(\gamma,p)$ is convex and $H_2(\gamma,p)$ is concave in $p \in \mathbb{R}^n$, $\forall \gamma \in \mathbb{R}$;*
(A4) *$H_1(\gamma,p)$ and $H_2(\gamma,p)$ are positively homogeneous of degree 1 in $p \in \mathbb{R}^n$, i.e., $H_1(\gamma,\lambda p) = \lambda H_1(\gamma,p)$, $\forall \lambda \geq 0$.*

The LSC viscosity solution of (34) *is given by*

$$u(x,y) = \inf_{y\in\mathbb{R}^n} \left[f(y) \vee H_1^{\sharp}\left(\frac{x-y}{t}\right)\right], \tag{36}$$

and the USC viscosity solution of (35) *is*

$$u(x,y) = \sup_{y\in\mathbb{R}^n} \left[f(y) \wedge H_{2\sharp}\left(\frac{x-y}{t}\right)\right], \tag{37}$$

where the conjugate operators $H^{\sharp}$ and $H_{\sharp}$ are defined as

$$H^{\sharp}(q) = \inf\left\{\gamma \in \mathbb{R} \,:\, H(\gamma,p) \geq \langle p,q\rangle, \forall p \in \mathbb{R}^n\right\}, \tag{38}$$

$$H_{\sharp}(q) = \sup\left\{\gamma \in \mathbb{R} \,:\, H(\gamma,p) \leq \langle p,q\rangle, \forall p \in \mathbb{R}^n\right\}. \tag{39}$$

The simplest case of admissible (A1)-(A4) convex Hamiltonian corresponds to $H(\gamma,p) = \gamma\|p\|$ such that, using Cauchy-Schwartz inequality, one gets

$$H^{\sharp}(q) = \inf\left\{\gamma \in \mathbb{R} \,:\, \gamma\|p\| \geq \langle p,q\rangle\right\} = \|q\|.$$

The associated concave Hamiltonian is given by $H(\gamma,p) = -\gamma\|p\|$, whose conjugate is also $H_{\sharp}(q) = \|q\|$. Using this case as a starting point, a prototype of PDE in the framework of operators in $(\max,\min)$-algebra can be defined

Definition 4. *Given any continuous and bounded function $f : E \to [a,b] \subset \mathbb{R}$, the canonic (Hamilton-Jacobi) PDE in $(\max,\min)$-morphology is defined as*

$$\begin{cases} \frac{\partial u}{\partial t} = \pm u\|\nabla u\|, \; x \in E, \;\; t > 0 \\ u(x,0) = f(x), \; x \in E \end{cases} \tag{40}$$

and its (unique weak) solutions at scale t are given by

$$u(x,t) = \sup_{y\in E}\left\{ f(y) \wedge \frac{\|x-y\|}{t}\right\} \quad \textit{(for + sign)}, \tag{41}$$

$$u(x,t) = \inf_{y\in E}\left\{ f(y) \vee \frac{\|x-y\|}{t}\right\} \quad \textit{(for − sign)}. \tag{42}$$

Therefore the viscosity solutions of Cauchy problem (40) are a supmin convolution and an infmax convolution using the conic structuring function $c_\lambda(x)$ given by (20), where the scale parameter is here the time; i.e., $\lambda = t$. More precisely, we note that these solutions

$$u(x,t) = (f \bigtriangledown (-c_t))(x) \;\; (\text{for} + \text{sign}),$$
$$u(x,t) = (f \bigtriangleup c_t)(x) \;\; (\text{for} - \text{sign}),$$

are not adjoint in the sense of Section 2, consequently their composition does not lead to opening or closing.

The model (40) can be generalized to

$$\frac{\partial u}{\partial t} = \pm \alpha u \|\nabla u\|, \quad x \in E, \; t > 0$$

with initial condition $u(x, 0) = f(x)$ and $\alpha > 0$, such that we easily see that the corresponding solutions are

$$\begin{aligned} u(x,t) &= (f \bigtriangledown (-c_{\alpha t}))(x) \quad (\text{for} + \text{sign}), \\ u(x,t) &= (f \bigtriangleup c_{\alpha t})(x) \quad (\text{for} - \text{sign}), \end{aligned}$$

or in other words, multiplying u by α involves a scaling in time by α. This principle can be a clue to explore the notion of *spatially adaptive* $(\max, \min)$-operators based on using a scale depending on space x, i.e., a model of the form $u_t = \pm\alpha(x)u\|\nabla u\|$.

5 Ubiquity of (max, min)-convolutions in mathematical morphology

It is obvious the connection between $(\max, \min)$-convolutions and the distance function or the flat morphology. We discuss now links to fuzzy morphology and to viscous morphology. Relationships of $(\max, \min)$-convolutions with Boolean random function characterization and geodesic dilation/erosion are discussed in [3].

Links with fuzzy morphology. The state-of-the-art on morphological operators based on fuzzy logic is very extensive, see for instance [7]. Results on fuzzy morphology discussed here are mainly based on Deng and Heijmans [8], see also [14].

In fuzzy logic, the two basic (Boolean) logic operators, the conjunction $C(s,t) = s \wedge t$ and the implication $I(s,t) = s \Rightarrow t \; (= \neg s \vee t)$, are extended from the Boolean domain $\{0,1\} \times \{0,1\}$ to the rectangle $[0,1] \times [0,1]$. A fuzzy conjunction is a mapping from $[0,1] \times [0,1]$ into $[0,1]$ which is increasing in both arguments and satisfies $C(0,0) = C(1,0) = C(0,1) = 0$ and $C(1,1) = 1$. A fuzzy implication is decreasing in the first argument, increasing in the second one and satisfies $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$.

Given a fuzzy set μ, the dilation and erosion by a fuzzy structuring element ν are then defined as [8]:

$$\delta_{\nu,C}(\mu)(x) = \sup_y \left\{ C\left(\nu(x-y), \mu(y)\right) \right\}, \tag{43}$$

$$\varepsilon_{\nu,C}(\mu)(x) = \inf_y \left\{ I\left(\nu(y-x), \mu(y)\right) \right\}. \tag{44}$$

As shown in [8], (I, C) is an adjunction if and only if $(\varepsilon_{\nu,C}, \delta_{\nu,C})$ is an adjunction. Two particular cases of conjunction and adjoint implication widely used in fuzzy logic are the Gödel-Brower:

$$C_{GB}(a,t) = \min(a,t); \quad I_{GB}(a,t) = \begin{cases} s, \; s < a \\ 1, \; s \geq a \end{cases} \tag{45}$$

and the Kleen-Dienes:

$$C_{KD}(a,t) = \begin{cases} 0, \; t \leq 1-a \\ t, \; t > 1-a \end{cases}; \quad I_{KD}(a,s) = \max(1-a, s) \tag{46}$$

It is consequently straightforward to see that the four operators that we have defined in Section 2 are just fuzzy dilations and erosions when they are applied to fuzzy sets (i.e., functions valued in $[0,1]$):

$$\begin{array}{ccc} \delta_{\nu,C_{GB}}(\mu)(x) = (\mu \bigtriangledown \nu)(x) & \overset{adjoint}{\longleftrightarrow} & \varepsilon_{\nu,C_{GB}}(\mu)(x) = (\mu \bigtriangleup^{*} \nu)(x), \\ \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \varepsilon_{\nu,C_{KD}}(\mu)(x) = (\mu \bigtriangleup \nu)(x) & \overset{adjoint}{\longleftrightarrow} & \delta_{\nu,C_{KD}}(\mu)(x) = (\mu \bigtriangledown^{*} \nu)(x). \end{array}$$

Links with viscous morphology. Theory and practice of morphological (flat) viscous operators was introduced by Vachier and Meyer [21,22]. The PDE formulation of these operators was done by Maragos and Vachier [15].

The idea of viscous operators is to apply a different scale (i.e., size) of structuring element at each upper level set. This principle can be seen now as an operator which locally adapts its activity with respect to the intensity. Let us formalize their definition according to [15]. For the sake of simplicity, let us consider a nonnegative bounded function $f : E \to [0, M]$. Viscous operators have been formulated as isotropic transforms, that is based on the use of balls B_λ as structuring elements.

Using intensity-adaptive operators and the two viscosity functions, two pairs of viscous dilation and erosion are defined for a given function f:

$$\delta_{\wedge}^{\text{visc}}(f) = \delta_{\lambda_{\wedge}(h)}(f) = \sup\left\{h \in [0,M] \; : \; x \in \left(X_h^{+}(f) \oplus B_{M-h}\right)\right\}, \tag{47}$$

$$\varepsilon_{\wedge}^{\text{visc}}(f) = \varepsilon_{\lambda_{\wedge}(h)}(f) = \sup\left\{h \in [0,M] \; : \; x \in \left(X_h^{+}(f) \ominus B_{M-h}\right)\right\}, \tag{48}$$

and

$$\delta_{\vee}^{\text{visc}}(f) = \delta_{\lambda_{\vee}(h)}(f) = \sup\left\{h \in [0,M] \; : \; x \in \left(X_h^{+}(f) \oplus B_{h}\right)\right\}, \tag{49}$$

$$\varepsilon_{\vee}^{\text{visc}}(f) = \varepsilon_{\lambda_{\vee}(h)}(f) = \sup\left\{h \in [0,M] \; : \; x \in \left(X_h^{+}(f) \ominus B_{h}\right)\right\}, \tag{50}$$

such that $\left(\varepsilon_{\wedge}^{\text{visc}}, \delta_{\wedge}^{\text{visc}}\right)$ and $\left(\varepsilon_{\vee}^{\text{visc}}, \delta_{\vee}^{\text{visc}}\right)$ form two adjunctions. The pairs $\left(\varepsilon_{\vee}^{\text{visc}}, \delta_{\wedge}^{\text{visc}}\right)$ and $\left(\varepsilon_{\wedge}^{\text{visc}}, \delta_{\vee}^{\text{visc}}\right)$ are dual by complement.

Let us introduce the following structuring function:

$$v(x) = \begin{cases} M - \|x\| & \text{if } \|x\| \leq M \\ 0 & \text{if } \|x\| > M \end{cases}$$

such that its complement structuring function is $v^c(x) = \|x\|$ if $\|x\| \leq M$ and M if $\|x\| > M$. We have $X_h^+(v) = B_{M-h}$ and $Y_h^-(v^c) = B_h$. Hence, viscous dilations and erosions (47)-(50) can be rewritten using the $(\max, \min)$-convolution (respectively expressions (12), (14), (18), (19)):

$$\begin{array}{ccc} \delta_{\wedge}^{\mathrm{visc}}(f)(c) = (f \bigtriangledown v)(x) & \overset{adjoint}{\longleftrightarrow} & \varepsilon_{\wedge}^{\mathrm{visc}}(f)(x) = (f \bigtriangleup^* v)(x), \\ \updownarrow \text{dual} & & \updownarrow \text{dual} \\ \varepsilon_{\vee}^{\mathrm{visc}}(f)(x) = (f \bigtriangleup v)(x) & \overset{adjoint}{\longleftrightarrow} & \delta_{\vee}^{\mathrm{visc}}(f)(x) = (f \bigtriangledown^* v)(x). \end{array}$$

In addition to the operator framework, a PDE formulation of viscous dilation and erosion was introduced in [15]. The proposed couple of PDEs are particular cases of the Hamilton-Jacobi models discussed above. More precisely, it corresponds to the case of the Hamiltonians given in expressions $H_1(\gamma, p) = (\alpha + \gamma)\|p\|$ and $H_2(\gamma, p) = -(\alpha + \gamma)\|p\|$, such that $H_1^\sharp(q) = H_{\sharp 2}(q) = \|q\| - \alpha$; or a pair $H_1(\gamma, p) = (\alpha - \gamma)\|p\|$ and $H_2(\gamma, p) = -(\alpha - \gamma)\|p\|$, with $H_1^\sharp(q) = H_{\sharp 2}(q) = \alpha - \|q\|$. Therefore solution $u(x,t)$ for $+$ sign of the PDE model is equivalent to viscous dilation $\delta_{\wedge}^{\mathrm{visc}}(f)$, but for $-$ sign it is not exactly equivalent to the viscous erosion $\varepsilon_{\wedge}^{\mathrm{visc}}(f)$. In our terminology, the latter is a case of adjoint infmax convolution while the solution for $-$ sign is an infmax convolution with the complemented structuring function.

6 Conclusion and Perspectives

Operators and filters underlying a formulation as $(\max, \min)$-convolutions are common in the state-of-the-art of mathematical morphology. However, their study *per se* has been neglected. From this epistemological viewpoint, we can conclude that the role of $(\max, \min)$-convolutions has been somewhat overshadowed by a multiplicity of viewpoints (fuzzy, viscous, "hitting of functions" in Choquet capacity, etc.) In order to address this theoretical lack, we have developed in our paper a rigorous formulation and characterization of the four convolution-like operators in $(\max, \min)$-algebra.

All the results on $(\max, \min)$-convolutions considered here are valid for functions supported in a general Banach space, consequently more general that the Euclidean space $\mathbb{R}^n$. In this generalization context, we plan to consider in particular the case of $(\max, \min)$-morphology for real-valued images on Riemannian manifolds.

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